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Stability of spiral flow between concentric circular cylinders at low axial Reynolds number

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The stability of a viscous liquid between two concentric rotating cylinders with an axial flow has been investigated. Attention has been confined to the case when the cylinders are rotating in the same direction, the gap between the cylinders is small and the axial flow is small. A perturbation theory valid in the limit when the axial Reynolds number $R \to 0$ has been developed and corrections have been obtained for Chandrasekhar's earlier results.

1. Introduction

The stability of viscous flow between rotating co-axial circular cylinders with an axial flow has been considered theoretically by Goldstein (1937), Chandrasekhar (1960, 1962), DiPrima (1960), Krueger & DiPrima (1963) and experimentally by Donnelly & Fultz (1960), Snyder (1962). While the experimental results are in general agreement with the theoretical predictions, there is some discrepancy. If the onset of instability occurs at a critical Taylor number T_c (based on the angular velocity of the inner cylinder), Snyder's experiment suggested a more rapid increase in T_c with small increase of the axial Reynolds number R from its zero value than that predicted by Chandrasekhar (1960), who considered an averaged axial velocity, and DiPrima (1960), who considered both the effects of an averaged axial velocity and of a parabolic axial-velocity profile. In an attempt to resolve this discrepancy, Chandrasekhar (1962) considered the case when the axial-velocity profile was parabolic and developed a perturbation theory which is valid in the limit $R \to 0$. He found that T_c increased much more rapidly than predicted by DiPrima. Recently Krueger & DiPrima (1963) have re-examined this problem, and their new results, while agreeing with the earlier results of DiPrima, do not predict the rapid initial increase of the critical Taylor number with R as obtained by Chandrasekhar (1962). They also suggested that, while the perturbation procedure used by Chandrasekhar was suitable, in the actual computation not enough terms in the series had been retained to give the correct coefficient of R^2.

In order to examine this difference, we have developed in § 2 a formal perturbation theory different from that of Chandrasekhar. Our results are in excellent agreement with those of DiPrima, and Krueger & DiPrima. Finally, in § 5 we have re-examined Chandrasekhar's perturbation series and have explained the reasons why his result is incorrect.

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2. The characteristic-value problem and the perturbation method

Assuming that the gap between the cylinders is small, and that the cylinders rotate in the same direction, the eigenvalue problem for marginal stability takes the form

$$\{D^2-a^2-i\beta+6iaRf(x)\}(D^2-a^2)u+12iaRu = -Ta^2v, \tag{1}$$

$$\{D^2-a^2-i\beta+6iaRf(x)\}v = u, \tag{2}$$

where $f(x) = \frac{1}{4}-x^2$, $D \equiv d/dx$, T is the Taylor number, R is the axial Reynolds number, a is a dimensionless wave-number, and β is the dimensionless time coefficient.† The boundary conditions are

$$u = Du = v = 0 \quad \text{at} \quad x = \pm\tfrac{1}{2}. \tag{3}$$

The homogeneous system of equations (1) and (2), together with the boundary conditions (3), define a characteristic value problem. The flow is unstable or stable according as the imaginary part of β is less than or greater than zero. We shall consider the neutrally stable case when the imaginary part of β is zero. Thus for given a and R we have to find a real β for which T is real. The critical Taylor number for a given R is then given by the minimum of the characteristic values T as a function of a.

We note that, when $R = 0$, equations (1) to (3) define the classical Taylor problem, which gives the critical Taylor number $T_c = 1707{\cdot}8$, with the critical wave-number $a_c = 3{\cdot}12$, and $\beta = 0$. Now, when R is small, we shall assume that

$$T = T_{00}+\epsilon T_{01}+\epsilon^2 T_{02}+\dots, \tag{4}$$

$$\beta = \epsilon\beta_1+\epsilon^2\beta_2+\dots, \tag{5}$$

$$u = W_0+\epsilon u_1+\epsilon^2 u_2+\dots, \tag{6}$$

$$v = \theta_0+\epsilon v_1+\epsilon^2 v_2+\dots, \tag{7}$$

where $\epsilon = 6R$, and W_0, θ_0 are the characteristic functions corresponding to the lowest characteristic value T_0 of the system of equations (1) and (2) when $R = 0$. Substituting for u, v, T, and β in equations (1) to (3), and equating coefficients of ϵ, ϵ^2 gives,

$$(D^2-a^2)^2 W_0 = -T_{00}a^2\theta_0, \tag{8}$$

$$(D^2-a^2)\theta_0 = W_0, \tag{9}$$

$$(D^2-a^2)^2 u_1 = -T_{00}a^2 v_1 - T_{01}a^2\theta_0 + iL_1 W_0, \tag{10}$$

$$(D^2-a^2)v_1 = u_1 + iM_1\theta_0, \tag{11}$$

and

$$(D^2-a^2)^2 u_2 = -T_{00}a^2 v_2 - T_{01}a^2 v_1 - T_{02}a^2\theta_0 + iL_1 u_1 + iL_2 W_0, \tag{12}$$

$$(D^2-a^2)v_2 = u_2 + iM_1 v_1 + i\beta_2\theta_0, \tag{13}$$

where

$$L_1 = \{\beta_1 - af(x)\}(D^2-a^2) - 2a,$$

$$M_1 = \beta_1 - af(x), \quad L_2 = \beta_2(D^2-a^2).$$

The above three systems of equations have to be considered together with the boundary conditions (3), where u and v have to be replaced by W_0 and θ_0, u_1 and v_1, u_2 and v_2, respectively.

† For the derivation of equations (1) and (2) see DiPrima (1960).

Now consider the eigenvalue problem for $\epsilon = 0$. We shall write the system of equations (1) to (3) for $\epsilon = 0$ as

$$(D^2-a^2)^2 W = -T_0 a^2 \theta, \tag{14}$$

$$(D^2-a^2)\theta = W, \tag{15}$$

with the boundary conditions

$$W = DW = \theta = 0 \quad \text{at} \quad x = \pm \tfrac{1}{2}. \tag{16}$$

It is known that the system of equations (14) to (16) is self-adjoint in the sense that there exists a relationship of duality between the proper solutions belonging to different characteristic values, i.e. if W_j, θ_j and W_k, θ_k are the proper solutions corresponding to the characteristic values T_j, T_k, respectively, then

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} W_j \theta_k \, dx = 0 \quad (j \neq k). \tag{17}$$

Besides, it can be shown (see, for example, DiPrima 1961) that the eigenvalues T_0 are real and positive. Further, the lowest eigenvalue T_{00} corresponds to even eigenfunctions W_0 and θ_0. Since the original system of equations (1) and (2) is even in x, and as we are interested in the perturbation of the lowest eigenvalue T_{00} corresponding to even functions W_0, θ_0, so we need only consider the even solutions for u_1, v_1, and u_2, v_2. Now the first three eigenvalues T_{00}, T_{10}, T_{20} of the system (14) to (16), corresponding to even eigenfunctions W_0, θ_0; W_1, θ_1; W_2, θ_2 respectively, for $a = 3{\cdot}12$ are

$$T_{00} = 1707{\cdot}8, \quad T_{10} = 172{,}362{\cdot}6, \quad T_{20} = 2{,}507{,}577{\cdot}0.$$

The eigenfunctions W_0 and θ_0 are given by Chandrasekhar (1961) and the functions W_1, θ_1, W_2, θ_2 are found easily by the formulas given by him (see Chandrasekhar 1961, pp. 36 to 39).

3. Solution

It has been shown by Krueger & DiPrima (1963) that for small values of R ($R \leqslant 5$) the critical wave-number a_c does not change from its value $3{\cdot}12$ for $R = 0$. So in the subsequent analysis we shall take $a = 3{\cdot}12$.

Now, to solve the system (10) and (11), we shall assume that u_1 and v_1 can be expanded in terms of the even proper solutions W_j and θ_j respectively of the system (14) and (15). Thus we write

$$u_1 = \sum_1^\infty A_j W_j, \quad v_1 = \sum_1^\infty B_j \theta_j. \tag{18}$$

The boundary conditions are automatically satisfied. Substituting for u_1 and v_1 in equations (10) and (11), multiplying (10) by W_j and (11) by θ_j and integrating with respect to x from $-\frac{1}{2}$ to $\frac{1}{2}$ we shall get

$$T_{01} = i\,\frac{\int W_0 L_1 W_0 \, dx - T_{00} a^2 \int \theta_0 M_1 \theta_0 \, dx}{a^2 \int W_0 \theta_0 \, dx}, \tag{19}$$

$$A_j = i\,\frac{\int W_j L_1 W_0 \, dx - T_{00} a^2 \int \theta_j M_1 \theta_0 \, dx}{a^2 (T_{j0} - T_{00}) \int W_j \theta_j \, dx} \quad (j \neq 0), \tag{20}$$

$$B_j = i\,\frac{\int W_j L_1 W_0\,dx - T_{j0}a^2\int\theta_j M_1\theta_0\,dx}{a^2(T_{j0}-T_{00})\int W_j\theta_j\,dx} \quad (j \neq 0), \tag{21}$$

$$A_0 - B_0 = i\,\frac{\int\theta_0 M_1\theta_0\,dx}{\int W_0\theta_0\,dx}, \tag{22}$$

where all the above and subsequent integrals have to be evaluated with respect to x from $-\frac{1}{2}$ to $\frac{1}{2}$.

From (19) it is obvious that T_{01} is imaginary. But since we are interested in real values of T, so T_{01} must equal to zero, i.e.

$$\int W_0 L_1 W_0\,dx - T_{00}a^2\int\theta_0 M_1\theta_0\,dx = 0, \tag{23}$$

which gives

$$\beta_1 = a\,\frac{\int W_0\{f(x)\,(D^2-a^2)+2\}\,W_0\,dx - T_{00}a^2\int f(x)\,\theta_0^2\,dx}{\int W_0(D^2-a^2)\,W_0\,dx - T_{00}a^2\int\theta_0^2\,dx}. \tag{24}$$

Similarly, to solve the system (12) and (13), we assume that

$$u_2 = \sum_1^\infty C_j W_j, \quad v_2 = \sum_1^\infty D_j\theta_j. \tag{25}$$

Then we obtain

$$T_{02} = -\frac{\left(\int W_0 L_1 W_0\,dx\right)\left(\int\theta_0 M_1\theta_0\,dx\right)}{a^2\{\int W_0\theta_0\,dx\}^2} + i\,\frac{\int W_0 L_2 W_0\,dx - \beta_2 T_{00}a^2\int\theta_0^2\,dx}{a^2\int W_0\theta_0\,dx}$$
$$+\,i\sum_{j\neq 0}\frac{A_j\int W_0 L_1 W_j\,dx - T_{00}a^2 B_j\int\theta_0 M_1\theta_j\,dx}{a^2\int W_0\theta_0\,dx}. \tag{26}$$

C_j and D_j can also be evaluated in the usual manner.

Now, since T has to be real, T_{02} must be real. So from (26), using (20) and (21), we conclude that T_{02} is real if and only if

$$\int W_0 L_2 W_0\,dx - \beta_2 T_{00}a^2\int\theta_0^2\,dx = 0.$$

This gives

$$\beta_2\int[(DW_0)^2 + a^2W_0^2 + T_{00}a^2\theta_0^2]\,dx = 0.$$

Hence $\beta_2 = 0$. Therefore

$$T_{02} = -\frac{\left(\int W_0 L_1 W_0\,dx\right)\left(\int\theta_0 M_1\theta_0\,dx\right)}{a^2\left(\int W_0\theta_0\,dx\right)^2} + i\sum_{j\neq 0}\frac{A_j\int W_0 L_1 W_j\,dx - T_{00}a^2 B_j\int\theta_0 M_1\theta_j\,dx}{a^2\int W_0\theta_0\,dx}, \tag{27}$$

where the A_j, B_j are given by (20), (21).† We may also write

$$T_{02} = i\,\frac{\int W_0 L_1 u_1\,dx - T_{00}a^2\int\theta_0 M_1 v_1\,dx}{a^2\int W_0\theta_0\,dx}. \tag{28}$$

This solves the problem of evaluating the lowest characteristic value T_0 of the system of equations (1) to (3), correct to the order of ϵ^2.

† We should note that from physical reasoning T should be an even function and β an odd function of R, which agrees with our results.

4. Numerical values

First, β_1 may be evaluated from the equation (24) by using the exact eigenfunctions W_0 and θ_0. We find
$$\beta_1 = 0\cdot608. \tag{29}$$
Next, to determine T_{02}, it is necessary to compute the series in (27). So we must know the eigenfunctions W_j, θ_j corresponding to T_{j0} ($j = 1, 2, \ldots$). We have computed the even eigenfunctions W_0, θ_0; W_1, θ_1 and W_2, θ_2. This is sufficient for our purposes since T_{j0} is rapidly increasing, and A_j and B_j are proportional to $(T_{j0} - T_{00})^{-1}$, for example $(T_{30} - T_{00})^{-1} \sim O(10^{-7})$. Thus truncating the series on the right-hand side of (27) after the second term we obtain
$$36T_{02} = 0\cdot9253 + 0\cdot3794 + 0\cdot0135 + \ldots = 1\cdot32. \tag{30}$$
We have also evaluated T_{02} and β_1 in another way. We have assumed $T_{01} = 0$ in the system of equations (10) and (11), and have solved the system for u_1 and v_1 numerically by the Runge–Kutta method using two step sizes, 0·05 and 0·02. Then we evaluated the integrals on the right of (28) by Simpson's rule. The results of these calculations are:
$$\beta_1 = 0\cdot608, \tag{31}$$
$$\left.\begin{aligned} 36T_{02} &= 1\cdot32 \quad \text{for} \quad h = 0\cdot02, \\ &= 1\cdot30 \quad \text{for} \quad h = 0\cdot05, \end{aligned}\right\} \tag{32}$$
where h is the step size. Hence we conclude that, correct to R^2,
$$\beta = 3\cdot65R, \quad T_c = 1707\cdot8 + 1\cdot32R^2,$$
$$a_c = 3\cdot12, \tag{33}$$
which are in excellent agreement with the values obtained by DiPrima (1960) and Krueger & DiPrima (1963). For example, they find, at $R = 0, 1, 2, 5$; $T_c = 1708, 1709, 1713, 1741$; $\beta = 0, 3\cdot65, 7\cdot30, 18\cdot25$, respectively, which can be seen to agree excellently with our result. On the other hand the result is considerably different from that obtained by Chandrasekhar (1962), who found
$$\beta = 3\cdot63R, \quad T_c = 1708 + 26\cdot5R^2 \quad \text{for} \quad a = 3\cdot1. \tag{34}$$
In the following section we have re-examined Chandrasekhar's solution and have found that, if we keep the terms of the relevant order, we get results close to ours.

5. Re-examination of Chandrasekhar's perturbation

Let us consider equations (1) to (3). Again u, v, T are expanded as in equations (6), (7) and (4), but we keep β fixed. Then β will be determined from the condition that T is real. Using equations (4), (6) and (7) in (1) to (3) we shall obtain three sets of equations for determining T_{00}, T_{01} and T_{02}, which are now complex-valued functions of β. Proceeding exactly as in the § 3† we can determine T_{00}, T_{01}, T_{02} as functions of β. Chandrasekhar determined only T_{00} and T_{01} for $a = 3\cdot1$ finding
$$T_{00}a^2 = (1\cdot6412 - 0\cdot002163\beta^2) \times 10^4 - 1\cdot267 \times 10^3 i\beta, \tag{35}$$
$$T_{01}a^2 = (7\cdot989\beta + 247\cdot3i)a. \tag{36}$$

† The orthogonality condition (17) still holds although w_j, θ_j are now complex-valued functions of β and x.

Now, if we neglect the terms containing ϵ^2 and higher powers of ϵ in (4) before we substitute values of T_{00} and T_{01} from (35) and (36) and impose the condition that T is real, we obtain equations (34).

However, as was pointed out by Krueger & DiPrima (1963), upon examining (4) we find that the expression for T_c would be correct to terms of $O(\epsilon^2)$, i.e. correct to terms of $O(R^2)$, if, and only if, T_{02} does not have a real part which is independent of β. If T_{02} has a real part independent of β, then clearly this will contribute to T correct to terms $O(R^2)$. To see if this could explain the discrepancy in Chandrasekhar's calculations, we have calculated both T_{01} and T_{02} when $\beta = 0$, and we find that, for $\beta = 0$, $T_{01}a = 247{\cdot}13i$, which checks with equation (36) and $36T_{02} = -24{\cdot}9$. Adding the correction from T_{02} in (34) we get, correct to $O(\epsilon^2)$,

$$\beta = 3{\cdot}63R, \quad T_c = 1708 + 1{\cdot}6R^2, \quad a = 3{\cdot}1. \tag{37}$$

Thus it would appear that, while the perturbation procedure suggested by Chandrasekhar is correct, considerable care must be taken in using it to ensure that at a certain order all terms have been evaluated.

In conclusion I express my deep gratitude to Professor R. C. DiPrima for suggesting this problem and for his many valuable comments and suggestions in the course of this work. I also wish to express my thanks to Mr A. Gross for help in programming the Runge–Kutta method and to Mr R. Grannick for checking some of the calculations. This work has been supported by the National Science Foundation grant GP-14 at the Mathematics Department, R.P.I.

Note added in proof. There is one important point in favour of doing the perturbation in the above way and that is that one can extend the perturbation series in a straightforward way to include higher powers of R. This is what one desires to have in a perturbation series. On the other hand an extension of Chandrasekhar's perturbation series to include higher powers of R becomes rather involved and complicated. In fact, it is prohibitive.

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